M2 Revision

2010

Mathematics Mechanics M2 Revision for WJEC.

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Projectiles

Basic Quotable Projectile Formulae

$$\begin{array}{ll} Vertical & Horizontal \\ a_y = \ddot{y} = -g & a_x = \ddot{x} = 0 \\ v_y = \dot{y} = u\sin\theta - gt & v_x = \dot{x} = u\cos\theta \\ s_y = y = u\sin\theta.t - \frac{1}{2}gt^2 & s_x = x = u\cos\theta.t \end{array}$$

Note: The number of dots over a variable indicates how many time it has been differentiated.

Deriving Projectile Formulae

Greatest Height

At the greatest height reached by the projectile, the vertical velocity is zero, $\dot{y} = 0$. Hence we can derive an equation for the time taken to reach the greatest height:

$$u\sin\theta - gt = 0 \rightarrow t = \frac{u\sin\theta}{g}$$

Subbing this into the vertical displacement equation gives the greatest height reached:

$$\begin{aligned} y &= u\sin\theta.\tfrac{u\sin\theta}{g} - \tfrac{1}{2}(u\sin\theta)^2 \\ &= \tfrac{u^2\sin^2\theta}{g}(1 - \tfrac{1}{2}) \\ &= \tfrac{u^2\sin^2\theta}{2g} \end{aligned}$$

Time of Flight

Method 1:

At the end of flight, the vertical displacement is zero, therefore:

$$\begin{aligned} y = 0 &\quad \rightarrow u\sin\theta.t - \tfrac{1}{2}gt^2 \\ &\quad t(u\sin\theta - \tfrac{gt^2}{2}) = 0 \\ \cancel{t = 0} &\quad , \quad t = \tfrac{2u\sin\theta}{g} \end{aligned}$$

Method 2:

Assuming no air resistance, given that the motion is parabolic, the time of flight is twice the time to the greatest height:

$$t_{total} = 2t_{maxheight} = 2 \times \frac{u\sin\theta}{g} = \frac{2u\sin\theta}{g}$$

Range

Range is the horizontal displacement at the end of flight, i.e. when the vertical displacement is zero. Subbing the time of flight equation into the horizontal displacement equation gives the range:

$$\begin{aligned} x &= u\cos\theta \times \frac{2\sin\theta}{g} \\ &= \frac{u^2 2\sin\theta\cos\theta}{g} \\ &= \frac{u^2\sin 2\theta}{g}\ (\because \sin 2A \equiv 2\sin A\cos A) \end{aligned}$$

Maximum Range

From the range equation we can infer (assuming no air resistance) that the maximum range for a fixed initial velocity (u) is achieved when $\sin 2\theta$ is maximised:

$$\sin 2\theta = 1 \rightarrow \theta = 45^\circ = \frac{\pi}{2}^c$$

Note: "c" is the symbol to denote a circular measure i.e. it's in radians

Worked Questions

Question (WJEC, 2010) Q. 5

1. The point A is at the top of a vertical cliff 39.2 m above sea level. A pebble is projected from point A with speed V ms^{-1} at angle of 30 degrees above the horizontal. The greatest height reached by the pebble is 4.9 m above A.
   a. Show that V = 19.6
   b. Calculate the time taken for the pebble to reach the surface of the sea.
   c. Find, correct to 3 significant figures, the speed of the pebble 3s after projection.

Answer

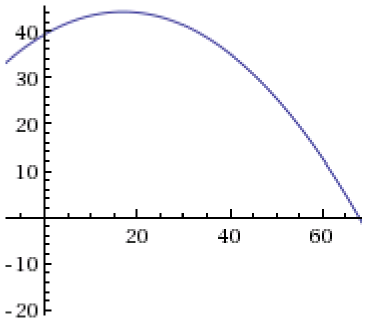


Figure 1 A plot of the motion of the curve

a. We know the greatest height is the height of the cliff plus 4.9 and that the vertical velocity will be 0 at this point.
   $\dot{y} = u\sin\theta.t - gt \quad \text{(Quotable)}$
   $0 = V\sin(30^\circ).t - gt$
   $V = 2g = 19.6$
b. The point in time when particle reaches the floor is when

   $y = -39.2$
   $y = u\sin\theta.t - \frac{1}{2}gt^2$
   $-39.2 = 19.6\sin(30^\circ).t - \frac{gt^2}{2}$
   $\frac{gt^2}{2} - gt - 4g = 0$ (19.6 and 39.2 are multiples of g)
   $t^2 - 2t - 8 = 0$ (multiplied the equation by $\frac{2}{g}$
   $(t+2)(t-4) = 0 \therefore t = 4\,\text{seconds}$
c. Just use the quotable equations for velocity then pythag to get resultant velocity:
   $\dot{y} = u\sin\theta - gt\ ,\ \dot{x} = u\cos\theta$
   $\dot{y} = 19.6\sin(30^\circ) - 3g = -2g\ ,\ \dot{x} = 19.6\cos(30^\circ) = \frac{\sqrt{3}g}{2}$
   $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{49\sqrt{19}}{10} = 21.4\,\text{s}\ (\text{to 3 s.f.})$

Particle Dynamics

This area covers a lot of things from strings obeying Hooke's Law, motion due to forces dependent on time, work done, power, and conservation of mechanical energy.

Hooke's Law

- A string (or spring) which length changes when acted on by a force is elastic.
- When no forces act on the string, it is at its natural length.
- Forces are balanced according to Newton III (Newton, 1687)
    - String exerts an inward pull (tension) which is equal to the extending force.
- It was discovered that:

$$T = \lambda \frac{x}{a}$$

Where:

- T is the tension in the string
- a is the natural length of the string
- x is the extension of the string
    - NOTE: not the extended length, but the difference between the new length and the natural length.
- λ is a constant called the "modulus of elasticity" which is dependent on the material.

Mechanical Energy

- The Principle of Conservation of Mechanical Energy is that at any point in a system mechanical energy will remain constant provided nothing is transferred to or from the system.
- There are three types of energy when considering mechanical energy:
    1. Kinetic Energy $E_k = \frac{1}{2}mv^2$
    2. Potential Energy $E_p = mg\Delta h$
    3. Elastic Potential Energy $E_e = \frac{\lambda x^2}{2a}$
- We can write the Principle of Conservation of Mechanical Energy mathematically for two points in a system, A and B, generally as:

$$(E_k + E_p + E_e)_A = (E_k + E_p + E_e)_B$$

Derivation of Elastic Potential Energy

Consider a string at length x_1 and x_2 with tensions T_1 and T_2 respectively; the string will be stretched from x_1 to x_2.

We know that work done is Force times change in displacement (distance moved): $W = F\Delta x$

The force, F, on average will be $F = \frac{1}{2}(T_1 + T_2)$

The change in displacement will be $\Delta x = (x_2 - x_1)$

Thus the work done in stretching the string is: $W = \frac{1}{2}(T_1 + T_2)(x_2 - x_1)$

If the string is initially unstretched then $T_1 = 0$ and $x_1 = 0$ which leaves us with $W = \frac{1}{2}Tx$

If we sub in Hooke's Law (Pg. 5) , $T = \lambda\frac{x}{a}$ we get $W = \frac{1}{2} \times \lambda\frac{x}{a} \times x$ which simplifies to:

$$W = \frac{\lambda x^2}{2a}$$

When a string is stretched, the work done is transferred to elastic potential energy so $E_p = W$.

Worked Questions

Question (WJEC, 2008) Q. 1

1. An elastic string, natural length 0.3m, supports a weight of 12N hanging freely in equilibrium. The total length of the string 0.55m.
    a. Calculate the modulus of elasticity in the string. [3]
    b. Find the elastic energy stored in the string. [3]

Answer

$a = 0.3\,\text{m}, W = 12\,\text{N}, x = 0.55 - 0.3 = 0.25\,\text{m}$

a. Using Hooke's Law (Pg. 5) $T = \lambda\frac{x}{a}$
   Resolving vertically, $T = W$
   $\lambda = \frac{Ta}{x} = \frac{12\times0.3}{0.25} = 14.4\,\text{N}$
b. Using the elastic potential energy formula derived on Pg. 5
   $E_p = \frac{\lambda x^2}{2a} = \frac{14.4\times(0.25)^2}{2\times0.3} = 1.5\,\text{J}$

Motion due to Force dependent on Time

- Questions on this topic will give the force acting on a particle as a function of time
  - i.e $F = f(t)$
- You will be required to use Newton II, $F = ma$
  - Calculus can be used if we rearrange for acceleration: $a = \frac{F}{m} = \frac{f(t)}{m}$ (see Rectilinear Motion on Pg. 11)

Worked Questions

Question (WJEC, 2008) Q. 3

1. A particle, of mass $5\,\text{kg}$ moves in a straight line under the action of a single force whos magnitude is $F\,\text{N}$ as time $t\,\text{s}$ is given by

$$F = 15t^2 - 60t\ ,\ t \geq 0$$

   a. Find the acceleration of the particle when $t = 2$
   b. The velocity of the particle at time $t\,\text{s}$ is denoted by $v\,\text{ms}^{-1}$. Given that $v = 35$ when $t = 0$, find an expression for v in terms of t.
   c. Calculate the least value of the speed of the particle.
   d. Determine the distance travelled by the particle between $t = 2$ and $t = 8$.

Answer

a. Applying Newton II, $F = ma$

$a = \frac{f(2)}{m} = \frac{15(2)^2 - 60(2)}{5} = -12\,\text{ms}^{-2}$

b. $a = \frac{f(t)}{m} = 3t^2 - 12t\ ,\ a = \frac{dv}{dt}$

$$\begin{aligned} dv &= a\ dt \\ \int_{35}^{v} dv &= \int_0^t (3t^2 - 12t)\ dt \\ [v]_{35}^{v} &= \left[t^3 - 6t^2\right]_0^t \\ v - 35 &= t^3 - 6t^2 \\ v &= t^3 - 6t^2 + 35 \end{aligned}$$

(don't put a C in, it's definite integration)

c. We have to find when the velocity is at a minimum, so we use the derivative which is the acceleration and set it to 0.

$3t^2 - 12t = 0$

$3t(t - 4) = 0$

So velocity has stationary points at $t = 0\ ,\ t = 4$

Subbing into the formula for velocity:

~~$v = 35\,\text{ms}^{-1}$~~ , $\underline{v = 3\,\text{ms}^{-1}}$

d. We are looking for distance travelled, not displacement so we must make sure that between 2 seconds and 8 seconds the particle is travelling forwards.

As $3\,\text{ms}^{-1}$ is the minimum velocity, the particle is always moving away from the origin.

$v = \frac{ds}{dt}$

$\int_0^s ds = \int_2^8 (t^3 - 6t^2 + 35)\ dt$

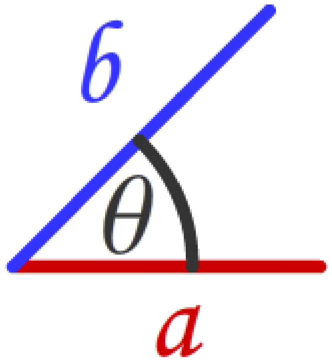
$s = \left[\frac{t^4}{4} - 2t^3 + 35t\right]_2^8$

$s = 222\,\text{m}$

Vectors

General Vector Properties

Dot/Scalar Product



The scalar or dot product is a way to multiply two vectors:

$$\mathbf{a}.\mathbf{b} = |a||b|\cos\theta$$

If a is perpendicular to b:

$$\mathbf{a}.\mathbf{b} = 0 \ \rightarrow \ \cos\theta = 0 \therefore \ \mathbf{a} \perp \mathbf{b}$$

If a is parallel to b:

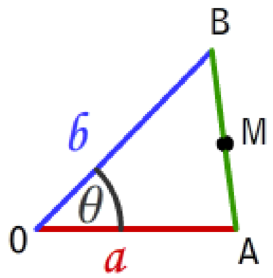
$$\mathbf{a}.\mathbf{b} = |a||b| \ \rightarrow \ \cos\theta = 1 \therefore \ \mathbf{a} \parallel \mathbf{b}$$

Vector Equation of a Line

The equation of a vector line can be represented in two ways:

$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) \ , \ \mathbf{r} = (1-t)\mathbf{a} + t\mathbf{b}$$

Midpoint of a Line



$$\begin{aligned} \vec{OM} &= \vec{OA} + \tfrac{1}{2}\vec{AB} \\ \vec{OM} &= \vec{OA} + \tfrac{1}{2}(\vec{OB} - \vec{BA}) \\ m_{AB} &= \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) \\ m_{AB} &= \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) \end{aligned}$$

Vector Calculus

Differentiating a displacement gives a velocity and differentiating a velocity gives acceleration.

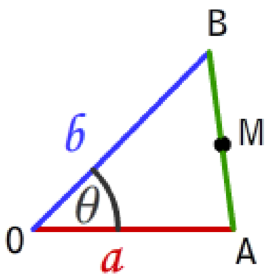
$$\begin{aligned} \mathbf{s} &= f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k} \\ \dot{\mathbf{s}} = \frac{\mathbf{ds}}{\mathbf{dt}} = \mathbf{v} &= f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k} \\ \ddot{\mathbf{s}} = \frac{\mathbf{dv}}{\mathbf{dt}} = \mathbf{a} &= f''(t)\hat{\imath} + g''(t)\hat{\jmath} + h''(t)\hat{k} \end{aligned}$$

Where $f(t)$, $g(t)$, $h(t)$ are functions of time e.g. $f(t) = t^3 + 5t^2 + 6t + 36$

Note: The hats on the i's, j's and k's mean indicate that they are unit vectors i.e. have length of 1 unit.

Conversely this process can be reversed by integration.

Relative Vectors



The relative displacement between 2 vectors can be found by considering the diagram to the right.

We see that the displacement between $\mathbf{b}$ and $\mathbf{a}$ is given by:

$$\mathbf{s}_{ab} = \mathbf{b} - \mathbf{a}$$

If the vectors are functions of time we can find for example when the two vectors are closest by differentiating the distance between the two functions to find a minimum point (C2 Coordinate Geometry).

The distance between two vectors is the modulus of their relative displacement and is found by using Pythagoras Theorem.

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ that have components that are functions of time:

$$\mathbf{a} = f(t)\hat{i} + g(t)\hat{j}\,,\ \mathbf{b} = p(t)\hat{i} + q(t)\hat{j}$$

Note: In the exam you are likely to be working in 3 dimensions but it's just another component ($\hat{k}$)

The distance between them at any time is given by:

$$|\,\mathbf{s}_{ab}| = \sqrt{(p(t) - f(t))^2 + (q(t) - g(t)^2)}$$

Now to find the time when there is a minimum distance we must differentiate $|\,\mathbf{s}_{ab}|$ and set it equal to zero but since we are setting it equal to zero anyway, it is quicker to just differentiate $|\,\mathbf{s}_{ab}|^2$ which gets rid of the root. So the time when they are closest together is given by:

$$\frac{d\,|\,\mathbf{s}_{ab}|^2}{dt} = \frac{d(p(t) - f(t))^2}{dt} + \frac{d(q(t) - g(t))^2}{dt} = 0$$

Which should solve to give at least one value of t. If there are more than solutions, disregard any where $t < 0$ then plug the values into the $|\,\mathbf{s}_{ab}|$ formula to find which gives the minimum distance.

Worked Questions

Question (WJEC, 2007) Q. 8

1. A toy plane A moving with constant velocity $(3\hat{\imath} - 2\hat{\jmath} + 5\hat{k})\,\mathrm{ms}^{-1}$ and at time $t = 0$, its position vector is $(3\hat{\jmath} - 140k)\,\mathrm{m}$. Another toy plane B is moving with constant velocity $(-2\hat{\imath} + 6\hat{\jmath} + 3\hat{k})\,\mathrm{ms}^{-1}$ and at time $t = 0$, its position vector is $(-9\hat{\imath} - 4\hat{\jmath} - 6k)\,\mathrm{m}$.
   a. Write down the position vectors of A and B at time $t\,\mathrm{s}$.
   b. Find an expression for the square of the distance between A and B at time $t\,\mathrm{s}$.
   c. Determine the time when A and B are closest together.

Answer

a. Simply add the position vectors to velocity multiplied by time:
$\mathbf{s}_a = 3t\hat{\imath} + (3 - 2t)\hat{\jmath} + (5t - 140)\hat{k}$
$s_b = (-9 - 2t)\hat{\imath} + (6t - 4)\hat{\jmath} + (3t - 6)\hat{k}$

b. Relative Displacement is given by:
$\mathbf{s}_{ab} = \mathbf{s}_b - \mathbf{s}_a$
$\mathbf{s}_{ab} = (-9 - 2t - 3t)\hat{\imath} + (6t - 4 - 3 + 2t)\hat{\jmath} + (3t - 6 - 5t + 140)\hat{k}$
$s_{ab} = (-9 - 5t)\hat{\imath} + (8t - 7)\hat{\jmath} + (134 - 2t)\hat{k}$
Distance between them is given by the square root of the sum of the squares of the components (See page 9). But we want the square of that, so it is just the sum of the squares of the components (i.e. just Pythagoras Theorem)
$|\,\mathbf{s}_{ab}|^2 = (-9 - 5t)^2 + (8t - 7)^2 + (134 - 2t)^2$
You can either expand this out and get the following which takes ages or just leave and do the chain rule in the next question.
$|\,\mathbf{s}_{ab}|^2 = 93t^2 - 558t + 18086$

c. We want the time at the minimum distance so we can just set the derivative of $|\,\mathbf{s}_{ab}|^2$ equal to zero (See why on page 9):

$$\begin{aligned}
\frac{d|\,\mathbf{s}_{ab}|^2}{dt} &= \frac{d(-9-5t)^2}{dt} + \frac{d(8t-7)^2}{dt} + \frac{d(134-2t)^2}{dt} \\
&= -10(-9 - 5t) + 16(8t - 7) - 4(134 - 2t) \\
&= 90 + 50t + 128t - 112 - 536 + 8t \\
&= 186t - 558 = 0 \\
t &= \frac{558}{186} = 3\,\mathrm{s}
\end{aligned}$$

Rectilinear Motion

For motion with acceleration that is not constant we use the following derivatives:

$$\begin{aligned} v &= \dot{x} = \tfrac{dx}{dt} & \\ a &= \ddot{x} = \tfrac{dv}{dt} & (v,t) \\ a &= \tfrac{d}{dt}(\tfrac{dx}{dt}) = \tfrac{d^2x}{dy^2} & (x,t) \\ a &= \tfrac{dx}{dt}\tfrac{dv}{dx} = v\tfrac{dv}{dx} & (x,v) \end{aligned}$$

We can also go the other way and use integration:

$$\begin{aligned} v &= \int a\,dt \\ s &= \int v\,dt \end{aligned}$$

Example 1

A particle P is projected from the origin Q so that it moves along the x-axis. At time t s after projection, the velocity of the particle, $v\,ms^{-1}$, is given by

$$v = 3t^2 - 24t + 45$$

(a) Show that P first comes to instantaneous rest when $t = 3$.
At rest $v = 0$

$$\begin{aligned} v = \quad & 3t^2 - 24t + 45 &= 0 \\ & t^2 - 8t + 15 &= 0 \\ & (t-3)(t-5) &= 0 \quad \therefore \text{First rest at t} = 3\text{s} \\ & t = 3\,,\ t = 5 & \end{aligned}$$

(b) Find and expression for the acceleration of P at time t s.

$$a \quad = \tfrac{dv}{dt} = 6t - 24$$

(c) Find an expression for the displacement of P from O at time t s.

$$\begin{aligned} x \quad &= \int v\,dt = \int(3t^2 - 24t + 45)dt \\ &= t^3 - 12t^2 + 45t + C \\ &\text{when } t = 0, s = 0 \rightarrow C = 0 \\ \therefore x \quad &= t^3 - 12t^2 + 45t \end{aligned}$$

(d) Find the distance travelled by the particle in the first 3 seconds of its motion.

$$x = 3^3 - 12 \times 3^2 + 45 \times 3 = 54\,m$$

(e) Find the distance travelled by the particle in the first 4 seconds of its motion.

$$\begin{aligned} x_{0\rightarrow 3} = 54 \quad & x_{0\rightarrow 4} = 4^3 - 12 \times 4^2 + 45 \times 4 = 52 \\ \therefore \quad & \text{distance travelled} = 54 + (54 - 52) = 56\,m \end{aligned}$$

Worked Questions

Question (WJEC, 2009) Q. 1

1. A particle moves along the x-axis and its velocity is $v\,\mathrm{ms}^{-1}$ at time $t\,\mathrm{s}$ is given by
$$v = \cos 2t - 3\sin t$$
    a. Find the acceleration of the body when $t = \pi$
    b. Give that $x = 4$ when $t = 0$, calculate the distance from the origin when $t = \frac{\pi}{4}$

Answer

a. $a = \frac{dv}{dt}$
$a = -2\sin 2t - 3\cos t$
When $t = \pi$, $a = 3\,\mathrm{ms}^{-2}$

b. $v = \frac{ds}{dt}$
$\int_4^s ds = \int_0^{\frac{\pi}{4}} (\cos 2t - 3\sin t)\, dt$
$[s]_4^s = \left[\frac{1}{2}\sin 2t + 3\cos t\right]_0^{\frac{\pi}{4}}$
$s - 4 = \frac{-5+3\sqrt{2}}{2}$
$s = \frac{3+3\sqrt{2}}{2}\,\mathrm{m}$

Circular Motion

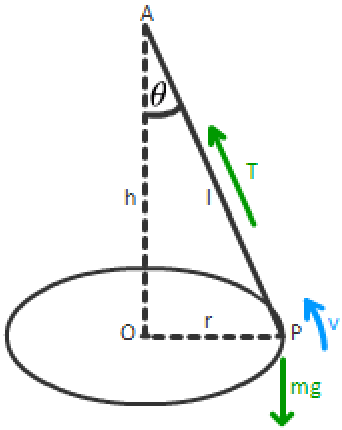
Basic Circular Motion

- A particle moving in a circle has a linear velocity and an angular velocity (typically with fixed magnitude)
  - The linear velocity (v) is how fast the particle travels around the circumference of the circle:
    $v = \frac{2\pi r}{T}$ where r is the radius of the circle and T is the time period – the time taken for the particle to make one revolution.
  - The angular velocity (ω) is how fast particle moves through an angle with respect to the origin. It can be found by dividing the amount of radians in one revolution (2π) by the time it takes to make one revolution (T):
    $\omega = \frac{2\pi}{T}$
  - Frequency (f) is the number of revolutions per second and is the reciprocal of the time period: $f = \frac{1}{T}$
- A particle moving in a circle has a changing velocity as its direction is changing (typically magnitude is constant though).
- Therefore it is being accelerated and must have a force acting on it.
- The force is called the "centripetal force" (goesy insy force) and always acts on the particle towards the centre of the circle.
  - Centripetal acceleration is given by: $a = \frac{v^2}{r} = \omega^2 r$
  - Subbing into Newton II ($F = ma$), force is given by: $F = \frac{mv^2}{r} = m\omega^2 r$

Summary of Formulae

Linear Velocity	$v = \frac{2\pi r}{T} = 2\pi f r = \omega r$
Angular Velocity	$\omega = \frac{2\pi}{T} = 2\pi f = \frac{v}{r}$
Angular Acceleration	$a = \frac{v^2}{r} = \omega^2 r$
Centripetal Force	$F = \frac{mv^2}{r} = m\omega^2 r$

Motion in a Horizontal Circle



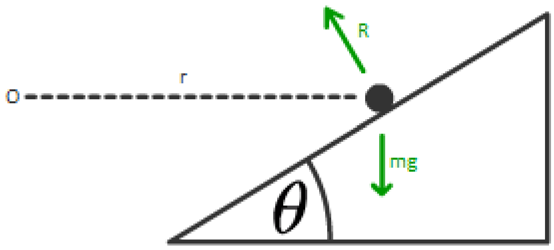
Conical Pendulum

Resolving Vertically: $T\cos\theta = mg$

Resolving Horizontally: $T\sin\theta = m\omega^2 r = \frac{mv^2}{r}$

Considering the triangle AOP: $r = l\sin\theta$, $h = l\cos\theta$

Banked Tracks



Resolving Vertically: $R\cos\theta = mg$

Resolving Horizontally: $R\sin\theta = m\omega^2 r = \frac{mv^2}{r}$

Dividing the equations:

$\frac{R\sin\theta}{R\cos\theta} = \frac{mv^2}{r} \div mg$

$\tan\theta = \frac{v^2}{rg} \rightarrow v = \sqrt{gr\tan\theta}$, $\theta = \arctan\frac{v^2}{rg}$

Worked Questions

Question (WJEC, 2009) Q. 7

1. A car, of mass $1000\,\mathrm{kg}$, is travelling in a horizontal circle of radius $250\,\mathrm{m}$ on a track which is banked at an angle α to the horizontal. When a car is travelling at $28\,\mathrm{ms}^{-1}$, it has no tendency to slip sideways, calculate the value of α.

Answer

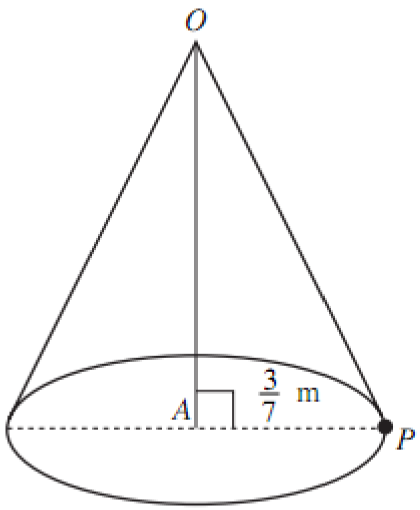
Resolving vertically: $R\cos\alpha = 1000g$

Resolving towards the centre (horizontally): $R\sin\alpha = \frac{mv^2}{r} = 3136$

Dividing the equations: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{3136}{1000g} \rightarrow \alpha = \arctan\frac{3136}{1000g} = 17.7°$

Question (WJEC, 2008) Q. 8

1. A particle P, of mass $4\,\mathrm{kg}$, is tied to one end of a light inextensible string and the other end of the string is fastened to a fixed point O. The particle P moves with a uniform speed of $2\,\mathrm{ms}^{-1}$ in a horizontal circle with centre A and radius $\frac{3}{7}\,\mathrm{m}$, as shown in the diagram.

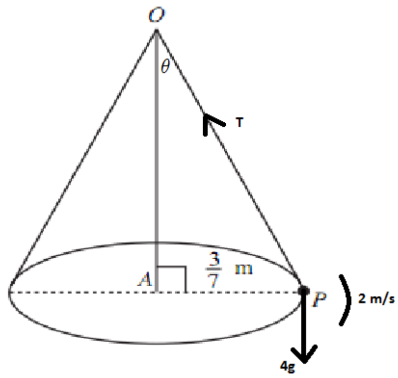


a. Find the size of $A\hat{O}P$.

b. Calculate the tension in the string.

c. Determine the length of the string.

Answer



a. Resolving ↑: $T\cos\theta = 4g$

Resolving ←: $T\sin\theta = \frac{mv^2}{r} = \frac{112}{3}$

Divide the equations:

$\frac{T\sin\theta}{T\cos\theta} = \frac{20}{21} \rightarrow \theta = \arctan\frac{20}{21} = 43.6^\circ$

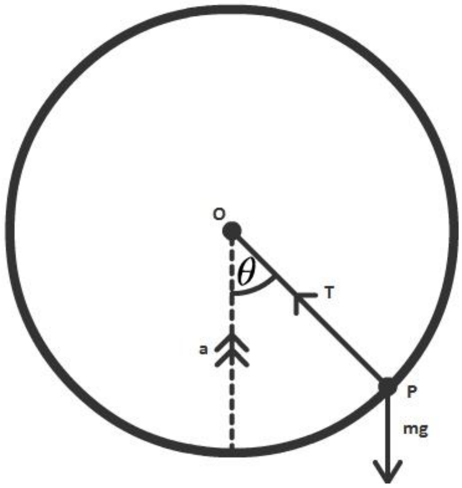
b. $T = \frac{4g}{\cos 43.6^\circ} = 54\,\text{N}$ (rounded)

c. Considering the triangle:

$\sin\theta = \frac{r}{l} \rightarrow l = \frac{3}{7}\sin 43.6^\circ = 0.3\,\text{m}$ (1d.p.)

Motion in a Vertical Circle

Typically most vertical circular motion requires resolving along PO where P is a general point of the particle on its circular path.



Energy considerations are also typical, see Mechanical Energy on page 5.

Resolving along PO: $T - mg\cos\theta = \frac{mv^2}{r}$

Using the Principle of Conservation of Mechanical Energy

$$(E_k + E_P)_A = (E_k + E_P)_P$$

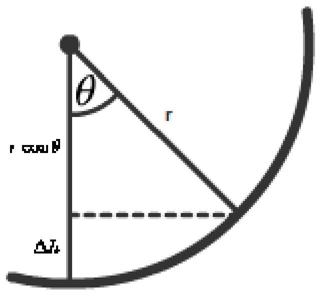
Let A represent the particle at its lowest point on the circle and the potential at this point be 0.

$$\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mg\Delta hn$$

$$v^2 = u^2 - 2g\Delta h$$

Where u is the initial speed of the particle, vny is the speed of the particle at the general point and Δh is the height above the lowest point on the circle.

Considering the triangle in the diagram:



$$\Delta h = r - r\cos\theta = r(1-\cos\theta)$$

$$\therefore v^2 = u^2 - 2gr(1-\cos\theta)$$

Complete Circles

Whether or not a particle will complete circles depends on the type of question you have.

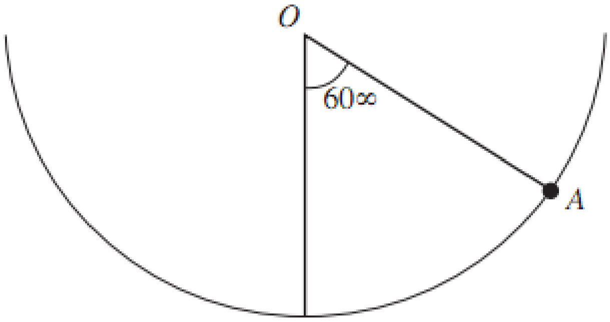
If the question is about a rigid rod, then you must make sure that at the highest point ($\theta = 180°$) the particle has a linear velocity greater than zero.
If you must find the angle when the particle is at its highest then use an equation relating v to θ, set $v = 0$ and solve for θ.

If you are given a question about a string or a particle rolling on the inside of the sphere then you must show that $T > 0$ (or $R > 0$) when $\theta = 180°$ so you will need an equation that relates tension or reaction force to θ. Again to find when the motion breaks down, set the force equal to zero and solve for θ.
The same is true for if the ball is on top of the circle (pigeon on St. Pauls) just take care with the direction of forces and where θ is measured from.

Worked Questions

Question (WJEC, 2008) Q. 9

1. A ball, of mass $2\,\mathrm{kg}$, is attached to one end of a light inextensible string of length $0.5\,\mathrm{m}$. The other end of the string is attached to a fixed point O. Initially, the ball is held at rest at a point A such that OA is inclined at an angle of $60°$ to the downward vertical through O.



The ball is projected downwards from A with velocity $4\,\mathrm{ms}^{-1}$ perpendicular to OA so that it starts describing a vertical circle centre O. When the string is inclined at an angle θ to the downward vertical, the speed of the ball is $v\,\mathrm{ms}^{-1}$.

a. Show that $v^2 = 9.8\cos\theta + 11.1$.

b. Find, in terms of θ, the tension of the string

Answer

a. Using the principle of conservation of mechanical energy:

$(E_k + E_p)_A = (E_k + E_p)_P$ [let the lowest point be zero potential]

$\frac{1}{2}mu^2 + mg(r - r\cos 60°) = \frac{1}{2}mv^2 + mg(r - r\cos\theta)$ [sub in all known variables]

$16 + 4.9 = v^2 + 9.8(1 - \cos\theta) = v^2 + 9.8 - 9.8\cos\theta$ [expand & tidy]

$v^2 = 9.8\cos\theta + 11.1$

b. Resolving along $\vec{A}$: $T - 2g\cos\theta = \frac{2v^2}{0.5} = 4v$

$T = 4v^2 + 2g\cos\theta$ [sub in v^2]

$T = 4(9.8\cos\theta + 11.1) + 2g\cos\theta$

$T = 58.8\cos\theta + 44.4t$

Definitions

A

Angular Velocity
A measure of how fast something is spinning ie. the rate of radians it moves through per second., 13

C

Centripetal Force
Always acts towards the centre. It is the force responsible for keeping a particle in circular motion e.g. Tension in a string, 13

Components
The components of a vector are its displacement in 2 more perpendicular directions usually denoted by i,j and k., 9

D

displacement
Shortest distance between 2 vectors, 9

Dot Product
Sum of the products of the components of two vectors. Also equal to the product the magnitudes of the vectors and the cosine of the angle between them., 8

H

Hooke's Law
The tension in a string or spring is proportional to the mount it is extended or compressed by., 5

L

Linear Velocity
Refers to velocity in a straight line. With regards to circular motion it is the speed that is always tangential to the circle at the point the particle is on it., 13

M

Mechanical Energy
Refers to the sum of kinetic, potential and elastic energy., 5

N

Newton II
Newton's second law of motion that F is equal to mass times acceleration., 7

Newton III
For every action there is an equal and opposite reaction., 5

P

Principle of Conservation of Mechanical Energy
Mechanical energy remains constant in system if no energy is transferred to or from it., 5

S

Scalar Product
See Dot Product, 8

Helpful Links

1. Circular Motion - http://www.youtube.com/watch?v=Otmg0-knGtE
2. Pure math videos - http://patrickjmt.com/
3. WolframAlpha (check equations) – http://www.wolframalpha.com